Note: In this problem set, expressions in green cells match corresponding expressions in the text answers.

$\mathscr{F}_s$

2 - 11 Fourier Transforms by Integration

Find the Fourier transform of f(x) (without using Table III in Sec. 11.10)

3. f(x) $=\begin{cases} 1 & a < x < b \\ 0 & \text{otherwise} \end{cases}$

This will involve the transform definition, according to s.m., panel (6) on p. 523

$$\hat{f}(\mathrm{w}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\mathrm{w}x}\, dx$$

```
Clear["Global`*"]
```

$$\text{fw = Simplify}\left[\frac{1}{\sqrt{2\pi}} \int_a^b e^{-i\,w\,x}\, dx,\ \{\text{Assumptions} \to a,\ b \in \text{Reals}\}\right]$$

$$-\frac{i\left(e^{-i\,a\,w} - e^{-i\,b\,w}\right)}{\sqrt{2\pi}\ w}$$

The problem gave no dificulties; green agrees with text.

5. f(x) $=\begin{cases} e^x & -a < x < a \\ 0 & \text{otherwise} \end{cases}$

$$\text{fw = Simplify}\left[\frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^x\, e^{-i\,w\,x}\, dx,\ \{\text{Assumptions} \to -a,\ a \in \text{Reals}\}\right]$$

$$\frac{i\sqrt{\frac{2}{\pi}}\ \text{Sinh}[a - i\,a\,w]}{i + w}$$

$$\text{N}\left[\frac{i\sqrt{\frac{2}{\pi}}\ \text{Sinh}[a - i\,a\,w]}{i + w}\ /.\ \{a \to 2,\ w \to 3\}\right]$$

0.026231 + 0.917442 i

$$\text{N}\left[\frac{\left(e^{(1-i\,w)\,a} - e^{-(1-i\,w)\,a}\right)}{\sqrt{2\pi}\ (1 - i\,w)}\ /.\ \{a \to 2,\ w \to 3\}\right]$$

0.026231 + 0.917442 i

For my taste, Mathematica's version of the answer looks a little neater than text’s.

7. $f(x)=\begin{cases} e^x & 0<x<a \\ 0 & \text{otherwise} \end{cases}$

```
fw = Simplify[1/Sqrt[2 π] ∫_0^a e^x e^(-i w x) dx, {Assumptions → a ∈ Reals}]
```

$$-\frac{i\left(1-e^{a-i a w}\right)}{\sqrt{2\pi}\,(i+w)}$$

```
N[-(i (1 - e^(a-i a w)))/(Sqrt[2 π] (i + w)) /. {a → 2, w → 3}]
```

-0.00395345 + 0.811803 i

```
N[(e^(-i a w) (1 + i a w) - 1)/(Sqrt[2 π] w^2) /. {a → 2, w → 3}]   (*text answer*)
```

-0.0760793 + 0.267754 i

Houston, we have a problem.

```
N[1/Sqrt[2 π] (-1/10 - (3 i)/10) (1 - e^(2-6 i))]      (* Symbolab, with a=2, w=3 *)
```

-0.00395345 + 0.811803 i

Acting as referee, Symbolab gives the nod to Mathematica (yellow, pink), but it still can't be green, because the answer does not match the text's.

9. $f(x)=\begin{cases} |x| & -1<x<1 \\ 0 & \text{otherwise} \end{cases}$

```
fw = FullSimplify[1/Sqrt[2 π] ∫_-1^1 Abs[x] e^(-i w x) dx]
```

$$\frac{\sqrt{\frac{2}{\pi}}\,(-1+\mathrm{Cos}[w]+w\,\mathrm{Sin}[w])}{w^2}$$

The extra power of **`FullSimplify`** brushed away all imaginary atomics.

11. $f(x)=\begin{cases} -1 & -1<x<0 \\ 1 & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$

$$\text{fw} = \text{FullSimplify}\left[\frac{1}{\sqrt{2\pi}}\int_{-1}^{0} -e^{-i\,w\,x}\,dx + \frac{1}{\sqrt{2\pi}}\int_{0}^{1} e^{-i\,w\,x}\,dx\right]$$

$$\frac{i\sqrt{\frac{2}{\pi}}\,(-1+\text{Cos}[w])}{w}$$

Another case where **FullSimplify** reformats the result to match the text answer.

12 - 17 Use of Table III in Sec. 11.10. Other Methods.

13. Obtain $\mathcal{F}\left(e^{-x^2/2}\right)$ from Table III.

$$f = \frac{1}{\sqrt{2\,a}}\,e^{-w^2/(4\,a)}\ /.\ a \to \frac{1}{2}$$

$$e^{-\frac{w^2}{2}}$$

I found it necessary to consolidate the power denominator ((4 a)) carefully with parentheses to get the right answer.

15. In Table III obtain formula 1 from formula 2.

Convert $\frac{e^{-i\,b\,w} - e^{-i\,c\,w}}{i\,w\,\sqrt{2\,\pi}}$ with $\begin{cases} 1 & \text{if } b < x < c \\ 0 & \text{otherwise} \end{cases}$ to $\frac{\sqrt{2}\,\text{Sin}[b\,w]}{\sqrt{\pi}\,w}$ with $\begin{cases} 1 & -b < x < b \\ 0 & \text{otherwise} \end{cases}$

$$\text{ComplexExpand}\left[\text{Re}\left[\frac{e^{i\,b\,w} - e^{-i\,b\,w}}{i\,w\,\sqrt{2\,\pi}}\right]\right]$$

$$\frac{\sqrt{\frac{2}{\pi}}\,\text{Sin}[b\,w]}{w}$$

The answer appendix does not list an answer for this problem. The s.m. handles this problem, and does a little legerdemain with the form. (or should I say leger-domain ha-ha) Positing that the problem only makes sense if the function domains are equal, the s.m. proposes to change the sign on the first term in the numerator (to positive), and to change the c in the domain restriction to b (changing the second term in the numerator likewise). Conforming to these changes, the table form of Formula 1 is achieved. (The useful command pair **ComplexExpand[Re[]]** was dug up from section 6.4.)

18 - 25 Discrete Fourier Transform

19. Find the transform of a general signal $f = [f_1\ f_2\ f_3\ f_4]^T$ of four values.

```
Fourier[{f₁, f₂, f₃, f₄}]
```

Fourier::fftl : Argument {f₁, f₂, f₃, f₄} is not a non-empty list or rectangular array of numeric quantities. >>

```
Fourier[{f₁, f₂, f₃, f₄}]
```

The discrete Fourier transform which the function **Fourier** is designed to perform consists of numerical calculations; it does not extend to a symbolic representation.

```
m = {{1, 1, 1, 1}, {1, -i, -1, i}, {1, -1, 1, -1}, {1, i, -1, -i}}
v = {f1, f2, f3, f4}
m.v
```

```
{{1, 1, 1, 1}, {1, -i, -1, i}, {1, -1, 1, -1}, {1, i, -1, -i}}
```

```
{f1, f2, f3, f4}
```

```
{f1 + f2 + f3 + f4, f1 - i f2 - f3 + i f4, f1 - f2 + f3 - f4, f1 + i f2 - f3 - i f4}
```

By doing it the long way, the text answer is obtained.

21. Find the transform (the frequency spectrum) of a general signal of two values $[f_1\ f_2]^T$.

```
FourierMatrix[2]
```

$$\left\{\left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}, \left\{\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\}\right\}$$

The above cell shows the Fourier matrix for 2 values. For my purposes, I want only the numerators.

```
m = {{1, 1}, {1, -1}}
v = {f1, f2}
m.v
```

```
{{1, 1}, {1, -1}}
```

```
{f1, f2}
```

```
{f1 + f2, f1 - f2}
```

The green cell above agrees with the text answer. (Due to ineptitude, I could not calculate the coefficients correctly using the material on p. 530.) **FourierMatrix** showed me not only the matrix I wanted but also that Fourier matrices exist for integers not conforming to 2^n.

23. Show that for a signal of eight sample values, w=$e^{-i/4} = (1-i)\Big/\sqrt{2}$. Check by squaring.